A BANACH SPACE DETERMINED BY THE WEIL HEIGHT

DANIEL ALLCOCK AND JEFFREY D. VAALER

ABSTRACT. The absolute logarithmic Weil height is well defined on the quotient group $\overline{\mathbb{Q}}^{\times}/\operatorname{Tor}(\overline{\mathbb{Q}}^{\times})$ and induces a metric topology in this group. We show that the completion of this metric space is a Banach space over the field $\mathbb{R}$ of real numbers. We further show that this Banach space is isometrically isomorphic to a co-dimension one subspace of $L^1(Y, \mathcal{B}, \lambda)$, where Y is a certain totally disconnected, locally compact space, $\mathcal{B}$ is the σ-algebra of Borel subsets of Y, and λ is a certain measure satisfying an invariance property with respect to the absolute Galois group $\operatorname{Aut}(\overline{\mathbb{Q}}/\mathbb{Q})$.

1. Introduction

Let k be an algebraic number field of degree d over $\mathbb{Q}$, v a place of k and k_v the completion of k at v. We select two absolute values from the place v. The first is denoted by $\| \ \|_v$ and defined as follows:

(i) if $v|\infty$ then $\| \ \|_v$ is the unique absolute value on k_v that extends the usual absolute value on $\mathbb{Q}_\infty = \mathbb{R}$,
(ii) if $v|p$ then $\| \ \|_v$ is the unique absolute value on k_v that extends the usual p-adic absolute value on $\mathbb{Q}_p$.

The second absolute value is denoted by $| \ |_v$ and defined by $|x|_v = \|x\|_v^{d_v/d}$ for all x in k_v, where $d_v = [k_v : \mathbb{Q}_v]$ is the local degree. If $\alpha \neq 0$ is in k then these absolute values satisfy the product formula

$$\prod_v |\alpha|_v = 1. \tag{1.1}$$

Let $\overline{\mathbb{Q}}$ be an algebraic closure of $\mathbb{Q}$ and $\overline{\mathbb{Q}}^{\times}$ the multiplicative group of nonzero elements in $\overline{\mathbb{Q}}$. The *absolute, logarithmic Weil height* (or simply the *height*)

$$h : \overline{\mathbb{Q}}^{\times} \to [0, \infty)$$

is defined as follows. Let α be a nonzero algebraic number, we select an algebraic number field k containing α, and then

$$h(\alpha) = \sum_v \log^+ |\alpha|_v, \tag{1.2}$$

where the sum on the right of (1.2) is over all places v of k. It can be shown that $h(\alpha)$ is well defined because the right hand side of (1.2) does not depend on the

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field k. By combining (1.1) and (1.2) we obtain the useful identity

$$2h(\alpha) = \sum_v \bigl|\log |\alpha|_v\bigr|, \tag{1.3}$$

where $|\ |$ (an absolute value without a subscript) is the usual archimedean absolute value on $\mathbb{R}$.

Let $\mathrm{Tor}\bigl(\overline{\mathbb{Q}}^\times\bigr)$ denote the torsion subgroup of $\overline{\mathbb{Q}}^\times$ and write

$$\mathcal{G} = \overline{\mathbb{Q}}^\times / \mathrm{Tor}\bigl(\overline{\mathbb{Q}}^\times\bigr)$$

for the quotient group. If ζ is a point in $\mathrm{Tor}\bigl(\overline{\mathbb{Q}}^\times\bigr)$, then it is immediate from (1.2) that $h(\alpha) = h(\zeta\alpha)$ for all points α in $\overline{\mathbb{Q}}^\times$. Thus h is constant on each coset of the quotient group $\mathcal{G}$, and so we may regard the height as a map

$$h : \mathcal{G} \to [0, \infty).$$

The height has the following well known properties (see [1, Section 1.5]):

(i) $h(\alpha) = 0$ if and only if α is the identity element in $\mathcal{G}$,
(ii) $h(\alpha^{-1}) = h(\alpha)$ for all α in $\mathcal{G}$,
(iii) $h(\alpha\beta) \le h(\alpha) + h(\beta)$ for all α and β in $\mathcal{G}$.

These conditions imply that the map $(\alpha, \beta) \mapsto h(\alpha\beta^{-1})$ defines a metric on the group $\mathcal{G}$ and therefore induces a metric topology. Our objective in this paper is to determine the completion of $\mathcal{G}$ with respect to this metric.

Let r/s denote a rational number, where r and s are relatively prime integers and s is positive. If α is in $\overline{\mathbb{Q}}^\times$ and ζ_1 and ζ_2 are in $\mathrm{Tor}\bigl(\overline{\mathbb{Q}}^\times\bigr)$, then all roots of the two polynomial equations

$$x^s - (\zeta_1\alpha)^r = 0 \quad \text{and} \quad x^s - (\zeta_2\alpha)^r = 0$$

belong to the same coset in $\mathcal{G}$. If we write $\alpha^{r/s}$ for this coset, we find that

$$(r/s, \alpha) \mapsto \alpha^{r/s}$$

defines a scalar multiplication in the abelian group $\mathcal{G}$. This shows that $\mathcal{G}$ is a vector space (written multiplicatively) over the field $\mathbb{Q}$ of rational numbers. Moreover, we have (see [1, Lemma 1.5.18])

$$h\bigl(\alpha^{r/s}\bigr) = |r/s| h(\alpha). \tag{1.4}$$

Therefore the map $\alpha \mapsto h(\alpha)$ is a norm on the vector space $\mathcal{G}$ with respect to the usual archimedean absolute value $|\ |$ on its field $\mathbb{Q}$ of scalars. From these observations we conclude that the completion of $\mathcal{G}$ is a Banach space over the field $\mathbb{R}$ of real numbers. It remains now to give an explicit description of this Banach space.

Let Y denote the set of all places y of the field $\overline{\mathbb{Q}}$. Let $k \subseteq \overline{\mathbb{Q}}$ be an algebraic number field such that $k/\mathbb{Q}$ is a Galois extension. At each place v of k we write

$$Y(k, v) = \{y \in Y : y|v\} \tag{1.5}$$

for the subset of places of Y that lie over v. Clearly we can express Y as the disjoint union

$$Y = \bigcup_v Y(k, v), \tag{1.6}$$

where the union is over all places v of k. If y is a place in $Y(k, v)$ we select an absolute value $\|\ \|_y$ from y such that the restriction of $\|\ \|_y$ to k is equal to $\|\ \|_v$.

As the restriction of $\| \ \|_v$ to $\mathbb{Q}$ is one of the usual absolute values on $\mathbb{Q}$, it follows that this choice of the normalized absolute value $\| \ \|_y$ does not depend on k.

In section 2 we show that each subset $Y(k,v)$ can be expressed as an inverse limit of finite sets. This determines a totally disconnected, compact, Hausdorff topology in $Y(k,v)$. Then (1.6) implies that Y is a totally disconnected, locally compact, Hausdorff space. Again the topology in Y does not depend on the field k. We also show that the absolute Galois group $\mathrm{Aut}(\overline{\mathbb{Q}}/k)$ acts transitively and continuously on the elements of each compact, open subset $Y(k,v)$.

In section 4 we establish the existence of a regular measure λ, defined on the Borel subsets of Y, that is positive on open sets, finite on compact sets, and satisfies $\lambda(\tau E) = \lambda(E)$ for all automorphisms τ in $\mathrm{Aut}(\overline{\mathbb{Q}}/k)$ and all Borel subsets E of Y. The restriction of the measure λ to each subset $Y(k,v)$ is unique up to a positive multiplicative constant. We construct λ so that

$$\lambda\big(Y(k,v)\big) = \frac{[k_v : \mathbb{Q}_v]}{[k : \mathbb{Q}]} \tag{1.7}$$

for each Galois extension k of $\mathbb{Q}$ and each place v of k. It follows from our construction that λ does not depend on the number field k. In particular, if l is any finite, Galois extension of $\mathbb{Q}$, if w is place of l and

$$Y(l,w) = \{y \in Y : y|w\},$$

then

$$\lambda\big(Y(l,w)\big) = \frac{[l_w : \mathbb{Q}_w]}{[l : \mathbb{Q}]}.$$

Next we consider the real Banach space $L^1(Y, \mathcal{B}, \lambda)$, where $\mathcal{B}$ denotes the σ-algebra of Borel subsets of Y. Let

$$\mathcal{X} = \Big\{F \in L^1(Y, \mathcal{B}, \lambda) : \int_Y F(y)\ \mathrm{d}\lambda(y) = 0\Big\}, \tag{1.8}$$

so that $\mathcal{X}$ is a co-dimension one linear subspace of $L^1(Y, \mathcal{B}, \lambda)$. For each point α in $\mathcal{G}$ we define a map $f_\alpha : Y \to \mathbb{R}$ by

$$f_\alpha(y) = \log \|\alpha\|_y. \tag{1.9}$$

If k is a finite Galois extension of $\mathbb{Q}$ that contains α, then $y \mapsto \log \|\alpha\|_y$ is constant on each compact, open set $Y(k,v)$. And the value of this map on each set $Y(k,v)$ is nonzero for only finitely many places v of k. It follows that $f_\alpha(y)$ is a continuous function on Y with compact support. Using (1.7) and the product formula (1.1), we find that

$$\begin{aligned}
\int_Y f_\alpha(y)\ \mathrm{d}\lambda(y) &= \sum_v \int_{Y(k,v)} \log \|\alpha\|_y\ \mathrm{d}\lambda(y) \\
&= \sum_v \frac{[k_v : \mathbb{Q}_v]}{[k : \mathbb{Q}]} \log \|\alpha\|_v \\
&= \sum_v \log |\alpha|_v = 0.
\end{aligned} \tag{1.10}$$

This shows that $\alpha \mapsto f_\alpha(y)$ maps $\mathcal{G}$ into the subspace $\mathcal{X}$. It follows easily that

$$f_{\alpha\beta}(y) = f_\alpha(y) + f_\beta(y) \quad \text{and} \quad f_{\alpha^{r/s}}(y) = (r/s) f_\alpha(y),$$

and therefore the map $\alpha \mapsto f_\alpha(y)$ is a linear map from the vector space $\mathcal{G}$ into $\mathcal{X}$. The L^1-norm of each function f_α is given by

$$\begin{aligned}\int_Y \bigl|f_\alpha(y)\bigr| \,\mathrm{d}\lambda(y) &= \sum_v \int_{Y(k,v)} \bigl|\log \|\alpha\|_y\bigr| \,\mathrm{d}\lambda_v(y)\\ &= \sum_v \frac{[k_v:\mathbb{Q}_v]}{[k:\mathbb{Q}]}\bigl|\log\|\alpha\|_v\bigr|\\ &= \sum_v \bigl|\log|\alpha|_v\bigr|\\ &= 2h(\alpha).\end{aligned} \tag{1.11}$$

This shows that the map $\alpha \mapsto f_\alpha$ is a linear isometry from the vector space $\mathcal{G}$ with norm determined by $2h$ into the subspace $\mathcal{X}$ with the L^1-norm. Let

$$\mathcal{F} = \bigl\{f_\alpha(y) : \alpha \in \mathcal{G}\bigr\} \tag{1.12}$$

denote the image of $\mathcal{G}$ under this linear map. Then $\alpha \to f_\alpha$ is a linear isometry from the vector space $\mathcal{G}$ (written multiplicatively) onto the vector space $\mathcal{F}$ (written additively). Now the completion of $\mathcal{G}$ is determined by finding the closure of $\mathcal{F}$ in $\mathcal{X}$.

Theorem 1. *Let $\mathcal{X}$ be the co-dimension one subspace of $L^1(Y,\mathcal{B},\lambda)$ defined by* (1.8)*. Then $\mathcal{F}$ is dense in $\mathcal{X}$.*

It is immediate from Theorem 1 that there exists an isometric isomorphism from the completion of the vector space $\mathcal{G}$ with respect to the height $2h$ onto the real Banach space $\mathcal{X}$.

The functions in the vector space $\mathcal{F}$ belong to the real vector space $C_c(Y)$ of continuous functions with compact support. Hence $\mathcal{F}$ belongs to the space $L^p(Y,\mathcal{B},\lambda)$ for $1 \le p \le \infty$. Theorem 1 asserts that the closure of $\mathcal{F}$ in $L^1(Y,\mathcal{B},\lambda)$ is the co-dimension one subspace $\mathcal{X}$. We also determine the closure of $\mathcal{F}$ with respect to the other L^p-norms.

Theorem 2. *If $1 < p < \infty$ then $\mathcal{F}$ is dense in $L^p(Y,\mathcal{B},\lambda)$.*

Let $C_0(Y)$ denote the Banach space of continuous real valued functions on Y which vanish at infinity equipped with the sup-norm. As $\mathcal{F} \subseteq C_c(Y) \subseteq C_0(Y)$, it is clear that the closure of $\mathcal{F}$ with respect to the sup-norm is a subspace of $C_0(Y)$.

Theorem 3. *The vector space $\mathcal{F}$ is dense in $C_0(Y)$.*

It follows from the classification of separable L^p-spaces (see [4, pp. 14-15]) that the Banach space $L^1(Y,\mathcal{B},\lambda)$ has a Schauder basis, or simply a basis. As $\mathcal{X} \subseteq L^1(Y,\mathcal{B},\lambda)$ is a closed subspace of co-dimension one, it is easy to show that $\mathcal{X}$ also has a basis. Then it follows from a well known result of Krein, Milman and Rutman [5] that a basis for $\mathcal{X}$ can be selected from the dense subset $\mathcal{F}$. Thus there exists a sequence of distinct elements $\alpha_1, \alpha_2, \dots$ in $\mathcal{G}$ such that the corresponding collection of functions

$$\bigl\{f_{\alpha_1}(y), f_{\alpha_2}(y), \dots\bigr\} \tag{1.13}$$

is a basis for the Banach space $\mathcal{X}$. That is, for every function F in $\mathcal{X}$ there exists a *unique* sequence of real numbers $x_1, x_2, \ldots$ such that

$$F(y) = \lim_{N\to\infty} \sum_{n=1}^{N} x_n f_{\alpha_n}(y)$$

in L^1-norm. While these remarks establish the existence of such a basis, it would be of interest to construct an explicit example of a sequence $\alpha_1, \alpha_2, \ldots$ in $\mathcal{G}$ such that the the corresponding sequence of functions (1.13) forms a basis for $\mathcal{X}$.

2. Preliminary Lemmas

We have stated Theorem 1 for the Weil height on algebraic number fields. However, many of the arguments can be given in the more general setting of a field K with a proper set of absolute values satisfying a product formula. We now describe this situation.

Let K be a field and let v be a place of K. That is, v is an equivalence class of nontrivial absolute values on K. We write K_v for the completion of K at the place v. If L/K is a finite extension of fields then there exist finitely many places w of L such that $w|v$. In general we have

$$\sum_{w|v} [L_w : K_v] \le [L : K],$$

where L_w is the completion of L at w. We say that v is *well behaved* if the identity

$$\sum_{w|v} [L_w : K_v] = [L : K]$$

holds for all finite extensions L/K, (see [6, Chapter 1, section 4]).

Let $\mathcal{M}_K$ be a collection of distinct places of K and at each place v in $\mathcal{M}_K$ let $\|\ \|_v$ denote an absolute value from v. We say that the collection of absolute values

$$\big\{\|\ \|_v : v \in \mathcal{M}_K\big\} \tag{2.1}$$

is *proper* if it satisfies the following conditions:

(i) each place v in $\mathcal{M}_K$ is well behaved,
(ii) if α is in $K^\times$ then $\|\alpha\|_v \ne 1$ for at most finitely many places v in $\mathcal{M}_K$,
(iii) if α is in $K^\times$ then the absolute values in (2.1) satisfy the product formula

$$\prod_{v\in\mathcal{M}_K} \|\alpha\|_v = 1.$$

Now suppose that (2.1) is a proper set of absolute values on K and L/K is a finite extension of fields. Let $\mathcal{M}_L$ be the collection of places of L that extend the places in $\mathcal{M}_K$. That is, if $W_v(L/K)$ is the finite set of places w of L such that $w|v$, then

$$\mathcal{M}_L = \bigcup_{v\in\mathcal{M}_K} W_v(L/K).$$

At each place w in $W_v(L/K)$ we select an absolute value $\|\ \|_w$ that extends the absolute value $\|\ \|_v$ on K. Then we define an equivalent absolute value $|\ |_w$ from the place w by setting

$$\log|\alpha|_w = \frac{[L_w : K_v]}{[L : K]} \log\|\alpha\|_w$$

for all α in $L^\times$. In general $\| \, \|_w$ and $| \, |_w$ are distinct but equivalent absolute values on L. And we note that $| \, |_w$ *is* an absolute value because

$$0 < \frac{[L_w : K_v]}{[L : K]} \le 1.$$

Then it follows, as in [6, Chapter 2, section 1], that

$$\big\{| \, |_w : w \in \mathcal{M}_L\big\} \tag{2.2}$$

is a proper set of absolute values on L. In particular, if α is in $L^\times$ then the absolute values in (2.2) satisfy the product formula

$$\prod_{w \in M_L} |\alpha|_w = 1.$$

We assume that $K \subseteq N$ are fields, that N/K is a (possibly infinite) Galois extension, and we write $\mathrm{Aut}(N/K)$ for the corresponding Galois group. We give $\mathrm{Aut}(N/K)$ the Krull topology, and we briefly recall how this is defined. Let $\mathcal{L}$ denote the set of intermediate fields L such that $K \subseteq L \subseteq N$ and L/K is a finite Galois extension. Obviously $\mathcal{L}$ is partially ordered by set inclusion. If L and M are in $\mathcal{L}$ then the composite field LM is in $\mathcal{L}$, $L \subseteq LM$, $M \subseteq LM$, and therefore $\mathcal{L}$ is a directed set. For each L in $\mathcal{L}$ let $\mathrm{Aut}(L/K)$ denote the Galois group of automorphisms of L that fix K. If $L \subseteq M$ are both in $\mathcal{L}$, we define $\pi_L^M : \mathrm{Aut}(M/K) \to \mathrm{Aut}(L/K)$ to be the map that restricts the domain of an automorphism in $\mathrm{Aut}(M/K)$ to the subfield L. Then each map π_L^M is a surjective homomorphism of groups and π_L^L is the identity map. It follows that

$$\big\{ \mathrm{Aut}(L/K), \pi_L^M \big\}$$

is an inverse system, and $\mathrm{Aut}(N/K)$ can be identified with the inverse (or projective) limit:

$$\mathrm{Aut}(N/K) = \varprojlim_{L \in \mathcal{L}} \mathrm{Aut}(L/K).$$

Thus $\mathrm{Aut}(N/K)$ is a profinite group, and therefore is a totally disconnected, compact, Hausdorff, topological group. We write

$$\pi_L : \mathrm{Aut}(N/K) \to \mathrm{Aut}(L/K)$$

for the canonical map associated with each L in $\mathcal{L}$. Then π_L is continuous and the collection of open sets

$$\big\{\pi_L^{-1}(\tau) : L \in \mathcal{L} \text{ and } \tau \in \mathrm{Aut}(L/K)\big\} \tag{2.3}$$

is a basis for the Krull topology in $\mathrm{Aut}(N/K)$.

Next we assume that v is a place of the field K. That is, v is an equivalence class of nontrivial absolute values on K. If L is in $\mathcal{L}$ we write $W_v(L/K)$ for the set of places w of L such that $w|v$. As L/K is a finite extension, it follows that $W_v(L/K)$ is a finite set. If $L \subseteq M$ belong to $\mathcal{L}$ we define connecting maps

$$\psi_L^M : W_v(M/K) \to W_v(L/K)$$

as follows: if w_M belongs to $W_v(M/K)$ then $\psi_L^M(w_M)$ is the unique place w_L in $W_v(L/K)$ such that $w_M|w_L$. If $L \subseteq M$ are in $\mathcal{L}$ then each absolute value on L extends to M and therefore each connecting map ψ_L^M is surjective. We give

each finite set $W_v(L/K)$ the discrete topology so that each map ψ_L^M is continuous. Clearly ψ_L^L is the identity map. We find that

$$\big\{W_v(L/K), \psi_L^M\big\}$$

is an inverse system of finite sets. Let

$$Y(K,v) = \varprojlim_{L \in \mathcal{L}} W_v(L/K)$$

denote the inverse limit and write $\psi_L : Y(K,v) \to W_v(L/K)$ for the canonical continuous map associated to each L in $\mathcal{L}$. It follows, as in [3, Appendix 2, section 2.4], that $Y(K,v)$ is a nonempty, totally disconnected, compact, Hausdorff space. Moreover, see [3, Appendix 2, section 2.3], the collection of open sets

$$\big\{\psi_L^{-1}(w) : L \in \mathcal{L} \text{ and } w \in W_v(L/K)\big\} \tag{2.4}$$

is a basis for the topology of $Y(K,v)$. Clearly each subset in the collection (2.4) is also compact, and for each field L in $\mathcal{L}$ we can write

$$Y(K,v) = \bigcup_{w \in W_v(L/K)} \psi_L^{-1}(w)$$

as a disjoint union of open and compact sets.

We recall that a map $g : Y(K,v) \to \mathbb{R}$ is *locally constant* if at each point y in $Y(K,v)$ there exists an open neighborhood of y on which g is constant.

Lemma 1. *Let $g : Y(K,v) \to \mathbb{R}$ be locally constant. Then there exists L in $\mathcal{L}$ such that for each place w in $W_v(L/K)$ the function g is constant on the set $\psi_L^{-1}(w)$.*

Proof. At each point y in $Y(K,v)$ there exists a field $L^{(y)}$ in $\mathcal{L}$ and a place $w^{(y)}$ in $W_v\big(L^{(y)}/K\big)$ such that y is contained in $\psi_{L^{(y)}}^{-1}\big(w^{(y)}\big)$ and g is constant on the open set $\psi_{L^{(y)}}^{-1}\big(w^{(y)}\big)$. By compactness there exists a finite collection of fields $L^{(1)}, L^{(2)}, \dots, L^{(J)}$ in $\mathcal{L}$, and for each integer j a corresponding place $w^{(j)}$ in $W_v\big(L^{(j)}/K\big)$, such that

$$Y(K,v) \subseteq \bigcup_{j=1}^{J} \psi_{L^{(j)}}^{-1}\big(w^{(j)}\big),$$

and g is constant on each open set $\psi_{L^{(j)}}^{-1}\big(w^{(j)}\big)$. Let $L = L^{(1)}L^{(2)} \cdots L^{(J)}$ be the composite field, which is obviously in $\mathcal{L}$. If w is a place of L then there exists an integer j such that

$$\psi_L^{-1}(w) \cap \psi_{L^{(j)}}^{-1}\big(w^{(j)}\big)$$

is not empty. As L is a finite extension of $L^{(j)}$, we conclude that $\psi_{L^{(j)}}^{L}(w) = w^{(j)}$, and therefore

$$\psi_L^{-1}(w) \subseteq \psi_{L^{(j)}}^{-1}\big(w^{(j)}\big). \tag{2.5}$$

Then (2.5) implies that g is constant on $\psi_L^{-1}(w)$. $\square$

Let $C\big(Y(K,v)\big)$ denote the real Banach algebra of continuous functions

$$F : Y(K,v) \to \mathbb{R}$$

with the supremum norm. Let $LC\big(Y(K,v)\big) \subseteq C\big(Y(K,v)\big)$ denote the subset of locally constant functions.

Lemma 2. *The subset $LC\big(Y(K,v)\big)$ is a dense subalgebra of $C\big(Y(K,v)\big)$.*

Proof. It is obvious that $LC\bigl(Y(K,v)\bigr)$ is a subalgebra of $C\bigl(Y(K,v)\bigr)$, and that $LC\bigl(Y(K,v)\bigr)$ contains the constant functions. Now suppose that y_1 and y_2 are distinct points in $Y(K,v)$. Let U_1 be an open neighborhood of y_1, and U_2 an open neighborhood of y_2, such that U_1 and U_2 are disjoint. Then there exists a field L in $\mathcal{L}$ and a place w in $W_v(L/K)$ such that

$$y_1 \in \psi_L^{-1}(w), \quad \text{and} \quad \psi_L^{-1}(w) \subseteq U_1.$$

As $\psi_L^{-1}(w)$ is both open and compact, the characteristic function of the set $\psi_L^{-1}(w)$ is a locally constant function that separates the points y_1 and y_2. Then it follows from the Stone-Weierstrass theorem that the subalgebra $LC\bigl(Y(K,v)\bigr)$ is dense in $C\bigl(Y(K,v)\bigr)$. □

We select an absolute value from the place v of K and denote it by $\|\ \|_v$. If L is in $\mathcal{L}$ and w is a place in $W_v(L/K)$, we select an absolute value $\|\ \|_w$ from w such that the restriction of $\|\ \|_w$ to K is equal to $\|\ \|_v$. As

$$N = \bigcup_{L\in\mathcal{L}} L,$$

it follows that each point (w_L) in $Y(K,v)$ determines a unique absolute value on the field N. That is, each point (w_L) in $Y(K,v)$ determines a unique place y of N such that $y|v$.

Now suppose y is a place of N such that $y|v$. Select an absolute value $\|\ \|_y$ from y such that the restriction of $\|\ \|_y$ to the subfield K is equal to $\|\ \|_v$. If L is in $\mathcal{L}$ then the restriction of $\|\ \|_y$ to L must equal $\|\ \|_{w_L}$ for a unique place w_L in $W_v(L/K)$. Thus each place y of N with $y|v$ determines a unique point (w_L) in the product

$$\prod_{L\in\mathcal{L}} W_v(L/K)$$

such that $y|w_L$ for each L. It is trivial to check that

$$\psi_L^M(w_M) = w_L$$

whenever $L \subseteq M$ are in $\mathcal{L}$. Therefore each place y of N with $y|v$ determines a unique point (w_L) in the inverse limit $Y(K,v)$. In view of these remarks we may identify $Y(K,v)$ with the set of all places y of N that lie over the place v of K. In this way we determine a totally disconnected, compact, Hausdorff, topology in the set of all places y of N that lie over the place v of K.

3. Galois action on places

Next we recall that the Galois group $\mathrm{Aut}(N/K)$ acts on the set $Y(K,v)$ of all places of N that lie over v. More precisely, if τ is in $\mathrm{Aut}(N/K)$ and y is in $Y(K,v)$, then the map

$$\alpha \mapsto \|\tau^{-1}\alpha\|_y \tag{3.1}$$

is an absolute value on N, and the restriction of this absolute value to K is clearly equal to $\|\ \|_v$. Therefore (3.1) determines a unique place τy in $Y(K,v)$. That is, the identity

$$\|\tau^{-1}\alpha\|_y = \|\alpha\|_{\tau y} \tag{3.2}$$

holds for all α in N, for all τ in $\mathrm{Aut}(N/K)$, and for all places y in $Y(K,v)$. It is immediate that $1y = y$ and $(\sigma\tau)y = \sigma(\tau y)$ for all σ and τ in $\mathrm{Aut}(N/K)$. Thus

$(\tau, y) \mapsto \tau y$ defines an action of the group $\mathrm{Aut}(N/K)$ on the set $Y(K, v)$. Moreover, $\mathrm{Aut}(N/K)$ acts transitively on $Y(K, v)$, (see [8, Chapter II, Proposition 9.1].)

Lemma 3. *The function* $(\tau, y) \mapsto \tau y$ *from* $\mathrm{Aut}(N/K) \times Y(K, v)$ *onto* $Y(K, v)$ *is continuous.*

Proof. Let L be in $\mathcal{L}$ and w in $W_v(L/K)$. In view of (2.4) we must show that

$$\big\{(\tau, y) \in \mathrm{Aut}(N/K) \times Y(K, v) : \tau y \in \psi_L^{-1}(w)\big\}$$

is open in $\mathrm{Aut}(N/K) \times Y(K, v)$ with the product topology. For w in $W_v(L/K)$ we define

$$E_w = \big\{(\sigma, z) \in \mathrm{Aut}(L/K) \times W_v(L/K) : \sigma z = w\big\}.$$

Then we have

$$\begin{aligned}
&\big\{(\tau, y) \in \mathrm{Aut}(N/K) \times Y(K, v) : \tau y \in \psi_L^{-1}(w)\big\} \\
&\qquad = \big\{(\tau, y) \in \mathrm{Aut}(N/K) \times Y(K, v) : \pi_L(\tau)\psi_L(y) = w\big\} \\
&\qquad = \bigcup_{(\sigma,z)\in E_w} \big\{(\tau, y) \in \mathrm{Aut}(K/k) \times Y(K, v) : \pi_L(\tau) = \sigma \text{ and } \psi_L(y) = z\big\} \\
&\qquad = \bigcup_{(\sigma,z)\in E_w} \pi_L^{-1}(\sigma) \times \psi_L^{-1}(z),
\end{aligned}$$

which is obviously an open subset of $\mathrm{Aut}(N/K) \times Y(K, v)$. $\square$

4. The invariant measure

In this section it will be convenient to write $G = \mathrm{Aut}(N/K)$. Let μ denote a Haar measure on the Borel subsets of the compact topological group G normalized so that $\mu(G) = 1$. If F is in $C\big(Y(K, v)\big)$ and z_1 is a point in $Y(K, v)$ then it follows from Lemma 3 that $\tau \mapsto F(\tau z_1)$ is a continuous function on G with values in $\mathbb{R}$. Let z_2 be a second point in $Y(K, v)$. Because G acts transitively on $Y(K, v)$, there exists η in G so that $\eta z_2 = z_1$. Then using the translation invariance of Haar measure we get

$$\int_G F(\tau z_1)\ \mathrm{d}\mu(\tau) = \int_G F(\tau\eta z_2)\ \mathrm{d}\mu(\tau) = \int_G F(\tau z_2)\ \mathrm{d}\mu(\tau). \tag{4.1}$$

It follows that the map $I_v : C\big(Y(K, v)\big) \to \mathbb{R}$ given by

$$I_v(F) = \int_G F(\tau z_v)\ \mathrm{d}\mu(\tau) \tag{4.2}$$

does not depend on the point z_v in $Y(K, v)$.

Let $\mathcal{M}_K$ be a collection of distinct places of K and at each place v in $\mathcal{M}_K$ let $\|\ \|_v$ denote an absolute value from v. We assume that

$$\big\{\|\ \|_v : v \in \mathcal{M}_K\big\}$$

is a proper collection of absolute values. Again we assume that N/K is a (possibly infinite) Galois extension of fields. Let Y be defined by the disjoint union

$$Y = \bigcup_{v\in\mathcal{M}_K} Y(K, v). \tag{4.3}$$

Thus Y is the collection of all places y of N such that $y|v$ for some place v in $\mathcal{M}_K$. It follows that Y is a nonempty, totally disconnected, locally compact, Hausdorff space.

Let $C_c(Y)$ denote the real vector space of continuous functions $F : Y \to \mathbb{R}$ having compact support. If F belongs to $C_c(Y)$ then there exists a finite subset $S_F \subseteq \mathcal{M}_K$ such that F is supported on the compact set

$$\bigcup_{v \in S_F} Y(K, v).$$

In particular we have $I_v(F) = 0$ for almost all places v of $\mathcal{M}_K$. Therefore we define $I : C_c(Y) \to \mathbb{R}$ by

$$I(F) = \sum_{v \in \mathcal{M}_K} \int_G F(\tau z_v) \, \mathrm{d}\mu(\tau), \tag{4.4}$$

where z_v is a point in $Y(K, v)$ for each place v in $\mathcal{M}_K$. By our previous remarks the value of each integral on the right of (4.4) does not depend on z_v, and only finitely many integrals on the right of (4.4) are nonzero. Hence there is no question of convergence in the sum on the right of (4.4).

Theorem 4. *There exists a σ-algebra $\mathcal{Y}$ of subsets of Y, that contains the σ-algebra $\mathcal{B}$ of Borel sets in Y, and a unique, regular measure λ defined on $\mathcal{Y}$, such that*

$$I(F) = \int_Y F(y) \, \mathrm{d}\lambda(y) \tag{4.5}$$

for all F in $C_c(Y)$. Moreover, the measure λ satisfies the following conditions:

(i) *If η is in G and F is in $L^1(Y, \mathcal{Y}, \lambda)$ then*

$$\int_{Y(K,v)} F(\eta y) \, \mathrm{d}\lambda(y) = \int_{Y(K,v)} F(y) \, \mathrm{d}\lambda(y) \tag{4.6}$$

at each place v in $\mathcal{M}_K$.

(ii) *If E is in $\mathcal{Y}$ then*

$$\lambda(E) = \inf\{\lambda(U) : E \subseteq U \subseteq Y \textit{ and } U \textit{ is open}\}.$$

(iii) *If E is in $\mathcal{Y}$ then*

$$\lambda(E) = \sup\{\lambda(V) : V \subseteq E \textit{ and } V \textit{ is compact}\}.$$

(iv) *If E is in $\mathcal{Y}$ and $\lambda(E) = 0$ then every subset of E is in $\mathcal{Y}$.*

Proof. Clearly (4.4) defines a positive linear functional on $C_c(Y)$. By the Riesz representation theorem, (see [9, Theorem 2.14 and Theorem 2.17]), there exists a σ-algebra $\mathcal{Y}$ of subsets of Y, containing the σ-algebra $\mathcal{B}$ of Borel sets in Y, and a regular measure λ defined on $\mathcal{Y}$, such that

$$I(F) = \int_Y F(y) \, \mathrm{d}\lambda(y) \tag{4.7}$$

for all F in $C_c(Y)$. If η is in G and F is in $C_c(Y)$, then by the translation invariance of the Haar measure μ we have

$$\begin{aligned} \int_{Y(K,v)} F(\eta y) \, \mathrm{d}\lambda(y) &= \int_G F(\eta \tau z) \, \mathrm{d}\mu(\tau) \\ &= \int_G F(\tau z) \, \mathrm{d}\mu(\tau) \\ &= \int_{Y(K,v)} F(y) \, \mathrm{d}\lambda(y) \end{aligned} \tag{4.8}$$

at each place v in $\mathcal{M}_K$. Initially (4.8) holds for all functions F in $C_c(Y)$. As $C_c(Y)$ is dense in the $L^1(Y, \mathcal{Y}, \lambda)$, (see [9, Theorem 3.14]), it follows in a standard manner that (4.8) holds also for functions F in $L^1(Y, \mathcal{Y}, \lambda)$.

The properties (ii), (iii) and (iv) attributed to λ are all consequences of the Riesz theorem. □

Because the Haar measure μ satisfies $\mu(G) = 1$, it is immediate from (4.2) and (4.5) that $\lambda\big(Y(K, v)\big) = 1$ at each place v in $\mathcal{M}_K$. As the places in $\mathcal{M}_K$ are well behaved, we obtain a further identity for the λ-measure of basic open sets in each subset $Y(K, v)$.

Theorem 5. *If L is in $\mathcal{L}$ and w is a place in $W_v(L/K)$, then*

$$\lambda\big(\psi_L^{-1}(w)\big) = \frac{[L_w : K_v]}{[L : K]}. \tag{4.9}$$

Proof. Let τ be in G. Then we have

$$\begin{aligned} \tau\psi_L^{-1}(w) &= \big\{\tau y \in Y(K, v) : \psi_L(y) = w\big\} \\ &= \big\{y \in Y(K, v) : \pi_L\big(\tau^{-1}\big)\psi_L(y) = w\big\} \\ &= \big\{y \in Y(K, v) : \psi_L(y) = \pi_L(\tau)w\big\} \\ &= \psi_L^{-1}\big(\pi_L(\tau)w\big). \end{aligned} \tag{4.10}$$

Now let w_1 and w_2 be distinct places in $W_v(L/K)$. Select τ in G so that $\pi_L(\tau)w_2 = w_1$. Then (4.10) implies that

$$\tau\psi_L^{-1}(w_2) = \psi_L^{-1}(w_1),$$

and using (4.6) we find that

$$\lambda\big\{\psi_L^{-1}(w_2)\big\} = \lambda\big\{\psi_L^{-1}(w_1)\big\}.$$

Because

$$Y(K, v) = \bigcup_{w \in W_v(L/K)} \psi_L^{-1}(w) \tag{4.11}$$

is a disjoint union of $|W_v(L/K)|$ distinct sets, the sets on the right of (4.11) all have equal λ-measure, and $\lambda(Y(K, v)) = 1$, we conclude that

$$\lambda\big(\psi_L^{-1}(w)\big) = |W_v(L/K)|^{-1}. \tag{4.12}$$

As v is well behaved we have

$$[L : K] = \sum_{w \in W_v(L/K)} [L_w : K_v]. \tag{4.13}$$

Because L/K is a Galois extension, all local degrees $[L_w : K_v]$ for w in $W_v(L/K)$ are equal, and we conclude from (4.13) that

$$|W_v(L/K)| = \frac{[L : K]}{[L_w : K_v]}. \tag{4.14}$$

The identity (4.9) follows now from (4.12) and (4.14). □

Let $LC_c(Y)$ be the algebra of locally constant, real valued functions on Y having compact support. Clearly we have $LC_c(Y) \subseteq C_c(Y)$.

Lemma 4. *Let g belong to $LC_c(Y)$. Then there exists L in $\mathcal{L}$ such that for each place w in $\mathcal{M}_L$ the function g is constant on the set $\psi_L^{-1}(w)$.*

Proof. Let $S_g \subset \mathcal{M}_K$ be a finite set of places of K such that the support of g is contained in the compact set

$$V_g = \bigcup_{v \in S_g} Y(K,v).$$

For each place v in S_g we apply Lemma 1 to the restriction of g to $Y(K,v)$. Thus there exists a field $L^{(v)}$ in $\mathcal{L}$ such that for each place w' in $W_v\big(L^{(v)}/K\big)$, the function g is constant on $\psi_{L^{(v)}}^{-1}\big(w'\big)$. Let L be the compositum of the finite collection of fields

$$\big\{L^{(v)} : v \in S_g\big\}.$$

Clearly L belongs to $\mathcal{L}$.

Let w be a place in $\mathcal{M}_L$. If $w|v$ and $v \notin S_g$, then g is identically zero on $\psi_L^{-1}(w)$, and in particular it is constant on this set. If $w|v$ and $v \in S_g$, then $w|w'$ for a unique place w' in $W_v\big(L^{(v)}/K\big)$. Because

$$\psi_L^{-1}(w) \subseteq \psi_{L^{(v)}}^{-1}\big(w'\big)$$

and g is constant on $\psi_{L^{(v)}}^{-1}\big(w'\big)$, it is obvious that g is constant on $\psi_L^{-1}(w)$. $\square$

Lemma 5. *For $1 \le p < \infty$ the set $LC_c(Y)$ is dense in $L^p(Y,\mathcal{B},\lambda)$. And $LC_c(Y)$ is dense in $C_0(Y)$ with respect to the* sup*-norm.*

Proof. Let $1 \le p < \infty$. Because $C_c(Y)$ is dense in $L^p(Y,\mathcal{B},\lambda)$, it suffices to show that if F is in $C_c(Y)$ and $\epsilon > 0$, then there exists a function g in $LC_c(Y)$ such that

$$\bigg\{\int_Y |F(y) - g(y)|^p \,\mathrm{d}\lambda(y)\bigg\}^{1/p} < \epsilon.$$

Let $S_F \subseteq \mathcal{M}_K$ be a nonempty, finite set of places such that F is supported on the compact set

$$V_F = \bigcup_{v \in S_F} Y(K,v).$$

For each v in S_F we apply Lemma 2 to the restriction of F to $Y(K,v)$. Thus there exists a locally constant function $g_v : Y(K,v) \to \mathbb{R}$ such that

$$\sup\big\{|F(y) - g_v(y)| : y \in Y(K,v)\big\} < |S_F|^{-1/p}\epsilon. \tag{4.15}$$

Now define $g : Y \to \mathbb{R}$ by

$$g(y) = \begin{cases} g_v(y) & \text{if } y \in Y(K,v) \text{ and } v \in S_F,\\ 0 & \text{if } y \in Y(K,v) \text{ and } v \notin S_F. \end{cases} \tag{4.16}$$

Then g is locally constant and supported on the compact set V_F. Therefore g belongs to $LC_c(Y)$. As $\lambda\big(Y(K,v)\big) = 1$ at each place v in $\mathcal{M}_K$, we get

$$\begin{aligned}\bigg\{\int_Y |F(y) - g(y)|^p \,\mathrm{d}\lambda(y)\bigg\}^{1/p} &= \bigg\{\sum_{v \in S_F} \int_{Y(K,v)} |F(y) - g_v(y)|^p \,\mathrm{d}\lambda(y)\bigg\}^{1/p}\\ &< \bigg\{\sum_{v \in S_F} |S_F|^{-1}\epsilon^p\bigg\}^{1/p} \le \epsilon.\end{aligned}$$

This proves the first assertion of the lemma.

As $C_c(Y)$ is dense in $C_0(Y)$ with respect to the sup-norm, the second assertion of the lemma follows by the same argument. In this case we select the locally constant functions $g_v : Y(K,v) \to \mathbb{R}$ so that

$$\sup\bigl\{|F(y) - g_v(y)| : y \in Y(K,v)\bigr\} < \epsilon.$$

Then we define $g : Y \to \mathbb{R}$ as in (4.16). Again we find that g belongs to $LC_c(Y)$, and the inequality

$$\sup\bigl\{|F(y) - g(y)| : y \in Y\bigr\} < \epsilon$$

is obvious. □

5. The completion of $\mathcal{G}$

In this section we return to the situation considered in the introduction. We let $K = \mathbb{Q}$, $N = \overline{\mathbb{Q}}$, and we let $\mathcal{M}_{\mathbb{Q}}$ be the set of all places of $\mathbb{Q}$. Then Y is the set of all places of $\overline{\mathbb{Q}}$, and Y is a nonempty, totally disconnected, locally compact, Hausdorff space. By Theorem 4 there exists a σ-algebra $\mathcal{Y}$ of subsets of Y, containing the σ-algebra $\mathcal{B}$ of Borel sets in Y, and a measure λ on $\mathcal{Y}$, satisfying the conclusions of that result. The basic identity (1.7) is verified by Theorem 5. Then the map

$$\alpha \to f_\alpha(y) \tag{5.1}$$

defined by (1.9) is a linear map from the $\mathbb{Q}$-vector space

$$\mathcal{G} = \overline{\mathbb{Q}}^{\times} / \operatorname{Tor}\bigl(\overline{\mathbb{Q}}^{\times}\bigr)$$

(written multiplicatively) into the vector space $C_c(Y)$. The identity (1.10) implies that each function $f_\alpha(y)$ belongs to the closed subspace $\mathcal{X} \subseteq L^1(Y, \mathcal{B}, \lambda)$ defined by (1.8). It follows from basic properties of the height, and in particular (1.4), that

$$\alpha \to 2h(\alpha),$$

defines a norm on $\mathcal{G}$ with respect to the usual archimedean absolute value on $\mathbb{Q}$. Then (1.11) shows that (5.1) defines a linear isometry of $\mathcal{G}$ into the subspace $\mathcal{X}$.

Lemma 6. *Let k be an algebraic number field and let $v \to t_v$ be a real valued function defined on the set of all places v of k. If*

$$\sum_v t_v \log|\alpha|_v = 0 \tag{5.2}$$

for all α in $k^{\times} / \operatorname{Tor}\bigl(k^{\times}\bigr)$, then the function $v \to t_v$ is constant.

Proof. Let S be a finite set of places of k containing all archimedian places, and assume that the cardinality of S is $s \geq 2$. We write $\mathbb{R}^s$ for the s-dimensional real vector space of column vectors $\boldsymbol{x} = (x_v)$ having rows indexed by places v in S. In particular, we write $\boldsymbol{t} = (t_v)$ for the column vector in $\mathbb{R}^s$ formed from the values of the function $v \to t_v$ restricted to S. And we write $\boldsymbol{u} = (u_v)$ for the column vector in $\mathbb{R}^s$ such that $u_v = 1$ for each v in S.

Let

$$U_S(k) = \{\eta \in k : |\eta|_v = 1 \text{ for all } v \notin S\}$$

denote the multiplicative group of S-units in k. By the S-unit theorem (stated as [7, Theorem 3.5]), there exist multiplicatively independent elements $\xi_1, \xi_2, \dots, \xi_{s-1}$ in $U_S(k)$ which form a fundamental system of S-units. Write

$$M = \bigl([k_v : \mathbb{Q}_v] \log \|\xi_r\|_v\bigr)$$

for the associated $(s-1)\times s$ real matrix, where $r = 1, 2, \ldots, s-1$ indexes rows and v in S indexes columns. As the S-regulator does not vanish, the matrix M has rank $(s-1)$. Hence the null space

$$\mathcal{N} = \big\{\boldsymbol{x} \in \mathbb{R}^s : M\boldsymbol{x} = \boldsymbol{0}\big\}$$

has dimension 1. From the product formula we have $M\boldsymbol{u} = \boldsymbol{0}$. Therefore $\mathcal{N}$ is spanned by the vector $\boldsymbol{u}$. By hypothesis we have $M\boldsymbol{t} = \boldsymbol{0}$, and it follows that $\boldsymbol{t}$ is a scalar multiple of $\boldsymbol{u}$. That is, the function $v \to t_v$ is constant on S. As S is arbitrary the lemma is proved. □

We now prove Theorem 1. Let $\mathcal{E}_1$ denote the closure of $\mathcal{F}$ in $\mathcal{X}$. As $\mathcal{F}$ is a vector space over the field $\mathbb{Q}$, it follows that $\mathcal{E}_1$ is a vector space over $\mathbb{R}$, and therefore $\mathcal{E}_1$ is a closed linear subspace of $\mathcal{X}$. If $\mathcal{E}_1$ is a proper subspace then it follows from the Hahn-Banach theorem (see [10, Theorem 3.5]) that there exists a continuous linear functional $\Phi : \mathcal{X} \to \mathbb{R}$ such that Φ vanishes on $\mathcal{E}_1$, but Φ is not the zero linear functional on $\mathcal{X}$. We will show that such a Φ does not exist, and therefore we must have $\mathcal{E}_1 = \mathcal{X}$.

Let $\Phi : \mathcal{X} \to \mathbb{R}$ be a continuous linear functional that vanishes on $\mathcal{E}_1$, but Φ is not the zero linear functional on $\mathcal{X}$. It follows from (1.8) that $\mathcal{X}^\perp \subseteq L^\infty(Y, \mathcal{B}, \lambda)$ is the one dimensional subspace spanned by the constant function 1. As the dual space $\mathcal{X}^*$ can be identified with the quotient space $L^\infty(Y, \mathcal{B}, \lambda)/\mathcal{X}^\perp$, there exists a function $\varphi(y)$ in $L^\infty(Y, \mathcal{B}, \lambda)$ such that $\varphi(y)$ and the constant function 1 are linearly independent, and

$$\Phi(F) = \int_Y F(y)\varphi(y)\ \mathrm{d}\lambda(y)$$

for all F in $\mathcal{X}$. Because Φ vanishes on $\mathcal{E}_1$ we have

$$\int_Y f_\alpha(y)\varphi(y)\ \mathrm{d}\lambda(y) = 0 \tag{5.3}$$

for each function f_α in $\mathcal{F}$.

Now let k be a number field in $\mathcal{L}$ and let α be in $k^\times/\operatorname{Tor}\big(k^\times\big) \subseteq \mathcal{G}$. From (4.9) and (5.3) we find that

$$\begin{aligned}
0 &= \sum_v \bigg\{\int_{\psi_k^{-1}(v)} \log\|\alpha\|_y \varphi(y)\ \mathrm{d}\lambda(y)\bigg\}\\
&= \sum_v \bigg\{\int_{\psi_k^{-1}(v)} \varphi(y)\ \mathrm{d}\lambda(y)\bigg\}\log\|\alpha\|_v\\
&= \sum_v \bigg\{\lambda\big(\psi_k^{-1}(v)\big)^{-1}\int_{\psi_k^{-1}(v)} \varphi(y)\ \mathrm{d}\lambda(y)\bigg\}\log|\alpha|_v.
\end{aligned} \tag{5.4}$$

It follows from Lemma 6 that the function

$$v \to \lambda\big(\psi_k^{-1}(v)\big)^{-1}\int_{\psi_k^{-1}(v)} \varphi(y)\ \mathrm{d}\lambda(y)$$

is constant on the set of places v of k. We write $c(k)$ for this constant.

Let $k \subseteq l$ be number fields in $\mathcal{L}$, and let v be a place of k. Using (4.9) and (4.14) we have

$$\lambda\big(\psi_k^{-1}(v)\big) = |W_v(l/k)|\lambda\big(\psi_l^{-1}(w)\big)$$

for all places w in the set $W_v(l/k)$. This leads to the identity

$$\begin{aligned} c(l) &= |W_v(l/k)|^{-1} \sum_{w\in W_v(l/k)} \Big\{\lambda\big(\psi_l^{-1}(w)\big)^{-1}\int_{\psi_l^{-1}(w)} \varphi(y)\ \mathrm{d}\lambda(y)\Big\} \\ &= \lambda\big(\psi_k^{-1}(v)\big)^{-1} \sum_{w\in W_v(l/k)} \Big\{\int_{\psi_l^{-1}(w)} \varphi(y)\ \mathrm{d}\lambda(y)\Big\} \\ &= \lambda\big(\psi_k^{-1}(v)\big)^{-1} \int_{\psi_k^{-1}(v)} \varphi(y)\ \mathrm{d}\lambda(y) \\ &= c(k). \end{aligned} \tag{5.5}$$

Thus there exists a real number C such that $C = c(k)$ for all fields k in $\mathcal{L}$.

Let g belong to $LC_c(Y)$. By Lemma 4 there exists a number field l in $\mathcal{L}$ such that g is constant on $\psi_l^{-1}(w)$ for each place w of l. Therefore we have

$$\begin{aligned} \int_Y g(y)\varphi(y)\ \mathrm{d}\lambda(y) &= \sum_w \Big\{\int_{\psi_l^{-1}(w)} g(y)\varphi(y)\ \mathrm{d}\lambda(y)\Big\} \\ &= C\sum_w \Big\{\lambda\big(\psi_l^{-1}(w)\big) g\big(\psi_l^{-1}(w)\big)\Big\} \\ &= C\sum_w \Big\{\int_{\psi_l^{-1}(w)} g(y)\ \mathrm{d}\lambda(y)\Big\} \\ &= C\int_Y g(y)\ \mathrm{d}\lambda(y). \end{aligned} \tag{5.6}$$

By Lemma 5 the set $LC_c(Y)$ is dense in $L^1(Y, \mathcal{B}, \lambda)$, and we conclude from (5.6) that

$$\int_Y F(y)\varphi(y)\ \mathrm{d}\lambda(y) = C\int_Y F(y)\ \mathrm{d}\lambda(y)$$

for all F in $L^1(Y, \mathcal{B}, \lambda)$. This shows that $\varphi(y) = C$ in $L^\infty(Y, \mathcal{B}, \lambda)$, and so contradicts our assumption that $\varphi(y)$ and the constant function 1 are linearly independent. Hence the continuous linear functional Φ does not exist, and therefore $\mathcal{E}_1 = \mathcal{X}$.

6. Proof of Theorem 2 and Theorem 3

We suppose that $1 < p < \infty$ and write $\mathcal{E}_p$ for the closure of $\mathcal{F}$ in $L^p(Y, \mathcal{B}, \lambda)$. As before, $\mathcal{E}_p$ is a closed linear subspace. By the Hahn-Banach theorem it suffices to show that if $\Phi : L^p(Y, \mathcal{B}, \lambda) \to \mathbb{R}$ is a continuous linear functional that vanishes on $\mathcal{E}_p$, then in fact Φ is identically zero on $L^p(Y, \mathcal{B}, \lambda)$.

Let $p^{-1} + q^{-1} = 1$, and let $\varphi(y)$ be an element of $L^q(Y, \mathcal{B}, \lambda)$ such that

$$\Phi(F) = \int_Y F(y)\varphi(y)\ \mathrm{d}\lambda(y)$$

for all F in $L^p(Y, \mathcal{B}, \lambda)$. We assume that Φ vanishes on $\mathcal{E}_p$, and then we have

$$\int_Y f_\alpha(y)\varphi(y)\ \mathrm{d}\lambda(y) = 0 \tag{6.1}$$

for each function f_α in $\mathcal{F}$.

Let k be a number field in $\mathcal{L}$ and let α be in $k^\times/\operatorname{Tor}(k^\times) \subseteq \mathcal{G}$. As before we apply (4.9) and (5.3) to obtain the identity (5.4). Then Lemma 6 implies that the function

$$v \to \lambda\bigl(\psi_k^{-1}(v)\bigr)^{-1} \int_{\psi_k^{-1}(v)} \varphi(y)\ \mathrm{d}\lambda(y) \tag{6.2}$$

is constant on the set of places v of k. Now, however, we apply Hölder's inequality and find that

$$\begin{aligned}\sum_v \left|\lambda\bigl(\psi_k^{-1}(v)\bigr)^{-1} \int_{\psi_k^{-1}(v)} \varphi(y)\ \mathrm{d}\lambda(y)\right|^q& \\ &\le \sum_v \left\{\lambda\bigl(\psi_k^{-1}(v)\bigr)^{-1} \int_{\psi_k^{-1}(v)} |\varphi(y)|^q\ \mathrm{d}\lambda(y)\right\} \\ &\le [k:\mathbb{Q}] \int_Y |\varphi(x)|^q\ \mathrm{d}\lambda(y) < \infty.\end{aligned}$$

This shows that the constant value of the function (6.2) is zero. Thus we have

$$\int_{\psi_k^{-1}(v)} \varphi(y)\ \mathrm{d}\lambda(y) = 0$$

for all k in $\mathcal{L}$ and for all places v of k. It follows using Lemma 4 that

$$\int_Y g(y)\varphi(y)\ \mathrm{d}\lambda(y) = 0$$

for all g in $LC_c(Y)$. By Lemma 5 the set $LC_c(Y)$ is dense in $L^p(Y, \mathcal{B}, \lambda)$, and we conclude that the continuous linear functional Φ is identically zero. This completes the proof of Theorem 2.

Next we suppose that $\mathcal{E}_\infty$ is the closure of $\mathcal{F}$ in $C_0(Y)$. Again it suffices to show that if $\Phi : C_0(Y) \to \mathbb{R}$ is a continuous linear functional that vanishes on $\mathcal{E}_\infty$, then Φ is identically zero on $C_0(Y)$. If Φ is such a linear functional, then by the Riesz representation theorem (see [9, Theorem 6.19]) there exists a regular signed measure ν, defined on the σ-algebra $\mathcal{B}$ of Borel sets in Y, such that

$$\Phi(F) = \int_Y F(y)\ \mathrm{d}\nu(y)$$

for all F in $C_0(Y)$. Moreover, we have $\|\Phi\| = \|\nu\|$, where $\|\Phi\|$ is the norm of the linear functional Φ and $\|\nu\|$ is the total variation of the signed measure ν. We assume that Φ vanishes on $\mathcal{E}_\infty$, and therefore

$$\int_Y f_\alpha(y)\ \mathrm{d}\nu(y) = 0$$

for each function f_α in $\mathcal{F}$. By arguing as in the proof of Theorem 2, we conclude that for each number field k in $\mathcal{L}$ the function

$$v \to \lambda\bigl(\psi_k^{-1}(v)\bigr)^{-1} \nu\bigl(\psi_k^{-1}(v)\bigr), \tag{6.3}$$

defined on the set of all places v of k, is constant. As

$$\begin{aligned}\sum_v \left|\lambda\big(\psi_k^{-1}(v)\big)^{-1}\nu\big(\psi_k^{-1}(v)\big)\right| \\ &\le [k:\mathbb{Q}]\sum_v \left|\nu\big(\psi_k^{-1}(v)\big)\right| \\ &\le [k:\mathbb{Q}]\|\nu\| < \infty,\end{aligned}$$

we conclude that the value of the constant function (6.3) is zero. This shows that

$$\nu\big(\psi_k^{-1}(v)\big) = 0$$

for all k in $\mathcal{L}$ and for all places v of k. It follows as before that

$$\Phi(g) = \int_Y g(y)\ \mathrm{d}\nu(y) = 0$$

for all g in $LC_c(Y)$. As $LC_c(Y)$ is dense in $C_0(Y)$ by Lemma 5, we find that Φ is identically zero on $C_0(Y)$. This proves Theorem 3.

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Department of Mathematics, University of Texas, Austin, Texas 78712 USA

E-mail address: `allcock@math.utexas.edu`

E-mail address: `vaaler@math.utexas.edu`